Droplet theory for Ising-like systems: two-loop results

1984 J. Phys. A: Math. Gen. 17 403

(http://iopscience.iop.org/0305-4470/17/2/025)

J. Phys. A: Math. Gen. **17** (1984) 403–417. Printed in Great Britain

Droplet theory for Ising-like systems: two-loop results

B Schmittmann

Department of Physics, University of Edinburgh, Edinburgh EH9 3JZ, UK

Received 2 June 1983

Abstract. A droplet theory for Ising-like systems valid in low spatial dimension can be formulated in terms of a surface tension Hamiltonian. We present an explicit two-loop calculation of the partition function for a single non-spherical droplet of one phase in a background of the other phase. The resulting β-function for the spherical interface is found to agree with the β-function of the planar interface. Corrections to the one-loop results are discussed.

1. Introduction

Droplet models are well known to provide essential insight into the physics of systems with coexisting phases. The 'classical' picture dates back as far as Becker and Döring (1935) and Cahn and Hilliard (1958, 1959): a review of these models and their refinements as well as further discussion can be found in Binder and Stauffer (1976).

More recently, a good picture has emerged for the subcritical region of the coexistence curve. There, the droplet theories of Andreev (1963) and Fisher (1967) were found capable of predicting an essential singularity of the free energy. Although this singularity is too weak to be experimentally observable, as was pointed out by Langer (1967), it is of considerable conceptual significance. It originates from the contributions of droplets which are large compared with the bulk correlation length ξ. Now large droplets are rather simple, because they are essentially spherical and occur with low probability, thus allowing for a dilute gas approximation. Further references are given in the reviews by Widom (1972), Domb (1976) and Binder (1976).

Within the critical region, however, the situation is far more complicated. Here, droplets of all scale sizes become important. Small droplets which now have to be considered are not necessarily spherical. One therefore has to include shape fluctuations. Also the dilute gas approximation becomes unjustified; in particular, as one encounters the phenomenon of droplet nesting, small droplets will sit inside larger droplets which are themselves embedded in even larger droplets. This typical scaling picture was recognised by Kadanoff (1976).

A theory incorporating these two features—droplet shape fluctuations and nesting—was proposed by Bruce and Wallace (1981, 1983, hereafter referred to as BW) for Ising-type systems in $d = 1 + \varepsilon$ dimensions. Their approach is based on a Hamiltonian given by the surface energy of a single non-spherical droplet of one phase in a background of the other phase. One can show by renormalisation group arguments

0305-4470/84/020403+15$02.25

that the partition function for such single droplet configurations gives the free energy of a multi-droplet ensemble, provided that the droplet *surfaces* are sufficiently dilute. This assumption can be justified in low spatial dimension.

The study of almost spherical surfaces is closely related to previous investigations of almost planar interfaces (Wallace and Zia 1979). In both cases, the critical behaviour of the interface is determined by large momentum, or, respectively, large angular momentum contributions, i.e. by short wavelength fluctuations. Consequently, noting that an almost spherical interface is locally indistinguishable from an almost flat interface, identical critical properties of the two models do not come as a surprise. In fact, both are designed to be representatives of the Ising universality class.

On this basis BW *assumed* that the two-loop β-function of the spherical interface would be identical to the corresponding β-function of the planar interface. In the absence of an explicit two-loop calculation for the spherical interface they had to use the planar result instead. This was necessary as only the *two-loop* β-function yields the correct *one-loop* amplitude for the partition function.

The present work closes this gap by giving the explicit two-loop renormalisation for the spherical interface. The assumption of BW is proved to be correct: the universal parts of the two β-functions do agree up to and including two-loop contributions.

Another motivation for this calculation lies in the observation that the study of two-loop graphs may reveal new structural features which remain invisible at the one-loop level. We shall indeed get a rather good picture of how divergences arise in this theory.

Furthermore, we obtain corrections to the partition function. In the limit of small scale sizes, this determines corrections to the exponent ratio β/ν. Naturally, this is rather of conceptual significance; one cannot expect that an extension of the results to two dimensions is very reliable numerically.

Technically the calculation will be interesting for a number of reasons. The expression for the surface of a non-spherical droplet (Günther *et al* 1980) involves angular momentum operators acting on a field which describes the deviation of the droplet from a spherical reference shape. It will therefore be convenient to expand the field in the eigenfunctions of the differential operator, i.e. in spherical harmonic functions. Associated with the shape fluctuations of the droplet are ultraviolet divergences which can be controlled in $1+\varepsilon$ dimensions. An elegant way to do so is the dimensional regularisation scheme proposed by 't Hooft and Veltman ('t Hooft and Veltman 1972, 't Hooft 1973), conventionally followed by renormalisation through minimal subtraction. Thus emerges the need for generalised spherical harmonics, i.e. functions which are homogeneous solutions of Laplace's equation in general dimension.

Techniques similar to the ones described in this paper were used by various authors. We mention only a selection: Drummond (1975) discusses dimensional regularisation on the sphere for ϕ^4 and ϕ^3 theories. McKane and Wallace (1978) and Drummond and Shore (1979) investigate ϕ^4 instantons in dimensional regularisation and are also led to a spherical formulation. These papers also contain further references.

In order to obtain the full two-loop expression, the partition function has to be evaluated up to and including $O(\varepsilon)$ and $O(T_0)$ terms in a double expansion with respect to ε and the coupling T_0.

Section 2 will deal with the $O(\varepsilon)$ terms. They are obtained by performing the functional integral over the *quadratic* part of the Hamiltonian, neglecting $O(\varepsilon^2)$. This is very closely related to the one-loop calculation by BW. The reader should therefore refer to their 1983 paper for a more detailed discussion.

In § 3 we shall evaluate the $O(T_0)$ contributions. To do so, we have to include interactions and analyse the possible two-loop graphs.

In § 4, we present the renormalisation and derive the β-function. We shall also discuss the corrections to the small R-limit of the partition function and its consequences. We conclude with a short summary of our results.

2. Gaussian contributions

The Ising-type droplet model is specified as follows. A 'droplet' is a connected volume of 'white' phase, say, in a background of 'black' phase. It is characterised by a certain scale size R, and its deviation from spherical in direction $\boldsymbol{\eta}$ is measured by a field $f(\boldsymbol{\eta})$.

The energy of this 'single droplet configuration' is simply the surface area of the droplet ($L_{ij}f = x_i\,\partial/\partial x_j - x_j\,\partial/\partial x_i$):

$$\mathcal{H} = \frac{1}{T_0}\int \mathrm{d}\Omega\,(R+f)^{d-1}[1+\tfrac{1}{2}(L_{ij}f)^2(R+f)^{-2}]^{1/2}. \tag{2.1}$$

Our task is to evaluate the single droplet partition function

$$Z_1 = \int \mathscr{D}f\,\mathrm{e}^{-\mathcal{H}[f]} \tag{2.2}$$

up to and including two-loop contributions. After expanding the field in spherical harmonics

$$f(\boldsymbol{\eta}) = \sum_{l,\alpha} a_{l,\alpha} Y_l^{\alpha}(\boldsymbol{\eta}) \tag{2.3}$$

and change of variables in the functional integral, (2.2) takes the form (see BW for a discussion of the measure)

$$Z_1 = \prod_{l,\alpha}\int \mathrm{d}a_{l,\alpha}\,\exp(-\mathcal{H}[a_{l,\alpha}]). \tag{2.4}$$

To extract the quadratic part $\mathcal{H}_0$ from the Hamiltonian, we expand $\mathcal{H}$ for $f/R \ll 1$ and find after partial integration ($\varepsilon = d-1$)

$$\mathcal{H}_0 = \tfrac{1}{2}\int \mathrm{d}\Omega\,(R^{\varepsilon-2}/T_0) f[\varepsilon(\varepsilon-1) - \tfrac{1}{2}L^2]f. \tag{2.5}$$

With the eigenvalue equation for the spherical harmonics

$$L^2 Y_l^{\alpha}(\boldsymbol{\eta}) = -2l(l+d-2)\,Y_l^{\alpha}(\boldsymbol{\eta}) \tag{2.6}$$

this gives the free propagator in angular momentum space:

$$G(l) = T_0 R^{2-\varepsilon}/[\varepsilon(\varepsilon-1) + l(l+\varepsilon-1)]. \tag{2.7}$$

Obviously, if $d = 1+\varepsilon$ and $\varepsilon \to 0$, the $l=0$ and $l=1$ modes would cause $G(l)$ to diverge and thus make a small ε expansion impossible. This problem can be overcome by excluding these modes from an expansion of f in spherical harmonics. To do so, one has to introduce collective coordinates. The $l=0$ mode, related to the 'breathing' of the droplet, can be absorbed into its scale size. This replaces the integration over a_{00} by an integration over all scale sizes. The Jacobian of the transformation is simply a

numerical factor. The $l=1$ mode describes rigid translations of the droplet. Its exclusion gives us an integral over all possible positions x_α of the droplet centre instead of the integral over $a_{1\alpha}$, and a non-trivial determinant:

$$\Delta = \exp\left[\operatorname{Tr}\ln\left(\delta_{ik}+\frac{d}{S_d}\int d\Omega\,(R+f)^{-1}\eta_i\eta_j L_{kj} f\right)\right]$$

$$= \exp\left[-\frac{1}{2}\left(\frac{d^2}{RS_d}\right)^2\int d\Omega\int d\Omega'\;\eta_i\eta_j\eta_i'\eta_j' f(\boldsymbol{\eta})f(\boldsymbol{\eta}')+O(f^3)\right]. \tag{2.8}$$

Thus we are left with

$$Z_1 = S_d^{1/2}\left(\frac{S_d}{d}\right)^{d/2}\int dR\prod_{i=1}^{d}\int dx_i\prod_{\substack{l,\alpha\\ l\geqslant 2}} e^{-\mathcal{H}}\Delta \tag{2.9}$$

where now

$$f(\boldsymbol{\eta}) = \sum_{\substack{l,\alpha\\ l\geqslant 2}} a_{l,\alpha} Y_l^\alpha(\boldsymbol{\eta}). \tag{2.9a}$$

With the Hamiltonian (2.1) and the propagator (2.6) an expansion in powers of T_0 is an expansion in the number of loops. More precisely, the quadratic part of $\mathcal{H}$ gives rise to corrections which are O(1) in T_0 and the interaction terms yield the higher T_0 orders. The quadratic part of the determinant, however, carries one T_0 power more than $\mathcal{H}_0$ and therefore contributes to the $O(T_0)$ corrections. Consequently in (2.8) we may safely neglect all higher-order contributions. In the following we shall evaluate the Gaussian part of our functional integral, keeping all $O(\varepsilon)$ and $O(T_0)$ corrections which arise at this stage. The treatment of the interaction terms will be postponed until § 3.

With the expansion (2.9*a*) $\mathcal{H}_0$ takes the form

$$\mathcal{H}_0 = \tfrac{1}{2}\sum_{\substack{l,\alpha\\ l\geqslant 2}}\frac{R^{\varepsilon-2}}{T_0}[\varepsilon(\varepsilon-1)+l(l+\varepsilon-1)]a_{l,\alpha}^2. \tag{2.10}$$

Further, (2.8) can be simplified if one recognises that only the $l=2$ modes of the field contribute to the integral. This can be seen by considering a particular representation of the spherical harmonics of degree 2:

$$Y_2^\alpha(\boldsymbol{\eta}) = u_{ij}^{(\alpha)}\eta_i\eta_j \tag{2.11}$$

where $u_{ij}^{(\alpha)}$ is a symmetric traceless tensor of rank 2. Note that this kind of representation is of course not limited to the case $l=2$.

The orthonormality of the Y_2^α's gives us the relation

$$u_{kl}^{(\alpha)}u_{kl}^{(\alpha')} = [d(d+2)/2S_d]\delta_{\alpha\alpha'}. \tag{2.12}$$

This result is sufficient for the subsequent calculations; an explicit basis is not required.

After inserting (2.11) into (2.9*a*) and (2.9*a*) into (2.8), using (2.12), one finds

$$\Delta = \exp[-d^3/S_d R^2(d+2)]a_{2a}^2. \tag{2.13}$$

Hence the evaluation of the Gaussian integral amounts to performing the sum

$$\Sigma = \sum_{l\geqslant 2}\nu_l(d)\ln\left(\frac{R^{\varepsilon-2}}{2\pi T_0}[l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)]+\frac{2d^3}{2\pi S_d R^2(d+2)}\delta_{l,2}\right) \tag{2.14}$$

where $\nu_l(d)$ denotes the degeneracy of the spherical harmonics of degree l in d dimensions:

$$\nu_l(d) = \frac{(d-1)}{\Gamma(d)} \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(l+1)}. \tag{2.15}$$

Rearranging (2.14) yields

$$\Sigma = \sum_{l\geqslant 2} \nu_l(d) \ln\left(\frac{R^{\varepsilon-2}}{2\pi T_0}[l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)]\right)$$
$$+ \nu_2(d) \ln\left(1+\frac{2d^3 T_0 R^{-\varepsilon}}{S_d(d+2)[\varepsilon(\varepsilon-1)+2(\varepsilon+1)]}\right). \tag{2.16}$$

We find that the corrections from the determinant, i.e. the second term in (2.16), are not only $O(T_0)$ but also pick up a factor ε from $\nu_2(d)$. This means that one can neglect them in a two-loop calculation. So finally we have

$$\Sigma = -(d+1)\ln\frac{R^{\varepsilon-2}}{2\pi T_0} + \frac{4\varepsilon}{\Gamma(d)}\left(\frac{1}{\varepsilon^2} - \gamma_1\right) - \frac{2\varepsilon}{\Gamma(d)} C + O(\varepsilon^2, \varepsilon T_0). \tag{2.17}$$

Here, γ_1 and C are constants given by

$$\gamma_1 = -\lim_{m\to\infty}\left(\sum_{l=1}^{m} \frac{\ln l}{l} - \frac{(\ln m)^2}{2}\right) = 0.072\,8166\ldots \tag{2.18a}$$

and

$$C = \sum_{l\geqslant 2} \frac{\ln l}{l(l^2-1)} = 0.2346\ldots. \tag{2.18b}$$

Hence, we have as resulting expression for the partition function at this stage

$$Z_1 = S_d^{(d+1)/2} d^{-d/2} \int \mathrm{d}R \int \mathrm{d}^d x \left(\frac{R^{\varepsilon-2}}{2\pi T_0}\right)^{(2+\varepsilon)/2}$$
$$\times \exp\left(-\frac{S_d R^{\varepsilon}}{T_0} - \frac{2}{\varepsilon\Gamma(d)} + \frac{\varepsilon}{\Gamma(d)}(C+2\gamma_1)\right)[1+O(T_0, \varepsilon^2, \varepsilon T_0)]. \tag{2.19}$$

Some remarks are in order. The first term in the exponential is the classical contribution which is caused by purely spherical droplets of radius R. It is followed by the corrections due to shape fluctuations. The one-loop corrections show a $1/\varepsilon$ divergence. To remove this singularity, one has to renormalise the dimensionless expansion parameter $T_0 R^{-\varepsilon}$. This will be discussed in § 4. We also get the first higher-order corrections, but they are simply of $O(\varepsilon)$. In § 3, we shall look at the more interesting contributions from interaction terms.

3. Two-loop graphs

3.1. Preliminaries

An inspection of the Hamiltonian (2.1) shows that we shall have to deal with cubic and quartic interactions, with and without derivatives. Hence there exist three distinct two-loop graphs for the partition function: the 'figure of eight', the 'London Transport' and the 'dumbbell' graph (figures 1(*a*)–(*c*)).

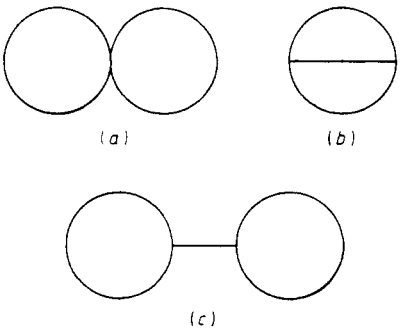

Figure 1. Two-loop graphs for the single droplet partition function.

In order to specify what an internal line or a vertex stands for we note the following. The standard Wick expansion for non-derivative interactions may be generalised to include derivative interactions. Thus all averages consisting of complicated products of fields and their derivatives can be split up into three basic contractions:

$$\langle f(\boldsymbol{\eta})f(\boldsymbol{\eta}')\rangle = T_0 R^{2-\varepsilon} \sum_{l\geq 2,\alpha} \frac{Y_l^\alpha(\boldsymbol{\eta})Y_l^\alpha(\boldsymbol{\eta}')}{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)}, \tag{3.1a}$$

$$\langle f(\boldsymbol{\eta})L_{ij}f(\boldsymbol{\eta}')\rangle = T_0 R^{2-\varepsilon} \sum_{l\geq 2,\alpha} \frac{Y_l^\alpha(\boldsymbol{\eta})L_{ij}Y_l^\alpha(\boldsymbol{\eta}')}{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)}, \tag{3.1b}$$

$$\langle L_{ij}f(\boldsymbol{\eta})L_{pq}f(\boldsymbol{\eta}')\rangle = T_0 R^{2-\varepsilon} \sum_{l\geq 2,\alpha} \frac{L_{ij}Y_l^\alpha(\boldsymbol{\eta})L_{pq}Y_l^\alpha(\boldsymbol{\eta}')}{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)}. \tag{3.1c}$$

Now let us consider a vertex produced by an interaction $gf^m[(L_{ij}f)^2]^n$. Of the $2n+m$ legs emanating from the vertex, m are labelled by spherical harmonics $Y_{l_1}^{\alpha_1}$, $Y_{l_2}^{\alpha_2}$, ... whereas the remaining $2n$ are labelled by $L_{ij}Y_{k_1}^{\beta_1}$, $L_{ij}Y_{k_2}^{\beta_2}$, $L_{pq}Y_{k_3}^{\beta_3}$, ... Then (3.1*a*) gives the propagator for a line connecting two 'non-derivative' legs. The other two equations denote the contractions of a non-derivative and a derivative and two derivative legs, respectively. The vertex itself carries a factor g, and there are also the normal combinatoric factors.

Let us now look at the various contributions.

3.2. Quartic interactions

From an expansion of $\mathcal{H}$ for small f we obtain three different quartic interactions:

$$\mathcal{H}_{\text{int}}^{(4)} = -\frac{R^{\varepsilon-4}}{T_0}\int \mathrm{d}\Omega \left[\tfrac{1}{32}(L_{ij}f)^2(L_{pq}f)^2 - \left(\frac{3}{4}-\frac{\varepsilon}{2}+\frac{\varepsilon(\varepsilon-1)}{8}\right)f^2(L_{ij}f)^2 - \binom{\varepsilon}{4}f^4\right]. \tag{3.2}$$

The 'figure of eight' represents the average of $\mathcal{H}_{\text{int}}^{(4)}$ with respect to $\mathcal{H}_0$:

$$\langle \mathcal{H}_{\text{int}}^{(4)}\rangle = \int \mathcal{D}f\, \mathcal{H}_{\text{int}}^{(4)}\, \mathrm{e}^{-\mathcal{H}_0} \Big/ \int \mathcal{D}f\, \mathrm{e}^{-\mathcal{H}_0}. \tag{3.3}$$

To demonstrate the principle, it will be sufficient to discuss the first term on the right-hand side of (3.2) in some detail. From the Wick expansion we find

$$\langle (L_{ij}f)^2(L_{pq}f)^2\rangle = \langle (L_{ij}f)^2\rangle^2 + 2\langle L_{ij}fL_{pq}f\rangle^2. \tag{3.4}$$

According to (3.1*c*), the first contribution on the right-hand side translates into

$$\int \mathrm{d}\Omega\, \langle (L_{ij}f)^2\rangle^2 = (T_0 R^{2-\varepsilon})^2 \int \mathrm{d}\Omega \sum_{l\geqslant 2,\alpha} \sum_{k\geqslant 2,\beta} \times \frac{L_{ij}Y_l^\alpha(\boldsymbol{\eta})L_{ij}Y_l^\alpha(\boldsymbol{\eta})}{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)} \frac{L_{pq}Y_k^\beta(\boldsymbol{\eta})L_{pq}Y_k^\beta(\boldsymbol{\eta})}{k(k+\varepsilon-1)+\varepsilon(\varepsilon-1)}$$

$$= 4(T_0 R^{2-\varepsilon})^2 \int \mathrm{d}\Omega \sum_{l\geqslant 2,\alpha} \sum_{k\geqslant 2,\beta} \frac{Y_l^\alpha l(l+\varepsilon-1)Y_l^\alpha}{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)} \times \frac{Y_k^\beta k(k+\varepsilon-1)Y_k^\beta}{k(k+\varepsilon-1)+\varepsilon(\varepsilon-1)} \tag{3.5}$$

after integrating by parts and exploiting the eigenvalue equation (2.6).

The sums over α and β can now be performed with the help of the addition theorem for spherical harmonics (see e.g. Erdélyi 1953):

$$\sum_{\alpha=1}^{\nu_l(d)} Y_l^\alpha(\boldsymbol{\eta})Y_l^\alpha(\boldsymbol{\eta}') = \frac{\nu_l(d)}{S_d} \frac{C_l^{(d-2)/2}(\boldsymbol{\eta}\cdot\boldsymbol{\eta}')}{C_l^{(d-2)/2}(1)}. \tag{3.6}$$

Here $C_l^{(d-2)/2}(t)$ denotes the Gegenbauer (or ultraspherical) polynomial of degree l and order $\frac{1}{2}(d-2)$. The normalisation of the Gegenbauers is such that

$$C_l^\lambda(1) = \Gamma(l+2\lambda)/\Gamma(2\lambda)\Gamma(l+1) \qquad (\lambda \neq 0).$$

Inserting this into (3.5) with $\boldsymbol{\eta} = \boldsymbol{\eta}'$ and performing the now trivial integration over $\mathrm{d}\Omega$ yields

$$\int \mathrm{d}\Omega\, \langle (L_{ij}f)^2\rangle^2 = 4(T_0 R^{2-\varepsilon})^2 S_d^{-1} \left[\sum_{l\geqslant 2} \nu_l(d)\right]^2 + \mathrm{O}(\varepsilon T_0^2). \tag{3.7}$$

The sum over degeneracies is easily evaluated (see BW), so finally,

$$\int \mathrm{d}\Omega\, \langle (L_{ij}f)^2\rangle^2 = 4(T_0 R^{2-\varepsilon})^2 S_d^{-1}(d+1)^2 + \mathrm{O}(\varepsilon T_0^2). \tag{3.8}$$

Note that this expression is O(1) in ε.

An attempt to calculate the second term in (3.4) in the same way fails, because identical angular momentum operators act on spherical harmonics of different degree. Thus it is not possible to generate a form similar to (3.5) by straightforward partial integration. However, one can make use of the symmetry properties of $\langle L_{ij}fL_{pq}f\rangle$. Averaging over all field configurations naturally eliminates all dependence on f, but the tensorial structure of the indices i, j and p, q is preserved: the result has to be antisymmetric under interchange of i and j, or p and q; and symmetric under the interchange of the *pairs* (i, j) and (p, q). Accordingly, it must be of the form

$$\langle L_{ij}f(\boldsymbol{\eta})L_{pq}f(\boldsymbol{\eta})\rangle = A(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) + B(\eta_i\eta_p\delta_{jq} - \eta_j\eta_p\delta_{iq} - \eta_i\eta_q\delta_{jp} + \eta_j\eta_q\delta_{ip}). \tag{3.9}$$

In order to determine the coefficients A and B, we apply (3.9) to two particular contractions, namely:

$$\langle (L_{ij}f(\boldsymbol{\eta}))^2\rangle = (d-1)(dA+2B) \qquad \text{and} \qquad \eta_i\eta_k\langle L_{ij}f(\boldsymbol{\eta})L_{kj}f(\boldsymbol{\eta})\rangle = (d-1)(A+B).$$

Moreover, inserting the definition of L_{ij}, one finds

$$\eta_i\eta_k\langle L_{ij}fL_{kj}f\rangle = \tfrac{1}{2}\langle (L_{ij}f)^2\rangle.$$

Hence

$$A=0, \qquad B=\tfrac{1}{2}\langle (L_{ij}f)^2\rangle/(d-1) \tag{3.10}$$

and

$$\langle L_{ij}fL_{pq}f\rangle^2=\langle (L_{ij}f)^2\rangle^2/(d-1). \tag{3.11}$$

This result is crucial: it exposes the kind of structure which leads to divergences in our theory. The simple contraction (3.8) is O(1) in ε, whereas one picks up a factor ε^{-1} from the 'mixed' contraction (3.11). Generally speaking divergences in more complicated graphs arise from the tensorial structure of contractions analogous to (3.11).

In the theory of the planar interface, the source of the divergences is similar. The planar analogue of $\langle L_{ij}f(\boldsymbol{\eta})L_{pq}f(\boldsymbol{\eta})\rangle^2$ is the contraction $\langle(\partial/\partial x_i)f(\boldsymbol{x})(\partial/\partial x_j)f(\boldsymbol{x})\rangle^2$. Here, $\boldsymbol{x}$ is a $(d-1)$-dimensional vector in the reference plane, and $f(\boldsymbol{x})$ is the deviation from planar in this point. The propagator is given by $G(q^2)=(q^2+m^2)^{-1}$ (see Wallace and Zia 1979, for details).

With the relation

$$\int \mathrm{d}^{d-1}q\, q_iq_iG(q^2)=\frac{\delta_{ij}}{(d-1)}\int \mathrm{d}^{d-1}q\, G(q^2)q^2$$

the planar analogue of (3.11) is

$$\int \mathrm{d}^{d-1}q \int \mathrm{d}^{d-1}k\, G(q^2)G(k^2)q_iq_jk_ik_j=\left(\int \mathrm{d}^{d-1}q\, q^2G(q^2)\right)^2\Big/(d-1).$$

It is easy to show that $\int \mathrm{d}^{d-1}q^2\, G(q^2)$ is O(1), so that we recover exactly the same structure as in (3.11).

Back to the spherical interface. With (3.7) and (3.11), (3.4) finally takes the form

$$\frac{R^{\varepsilon-4}}{32T_0}\int \mathrm{d}\Omega\,\langle (L_{ij}f)^2(L_{pq}f)^2\rangle=\frac{T_0R^{-\varepsilon}}{S_d}\left(\frac{1}{\varepsilon}+\frac{3}{2}\right)+\mathrm{O}(\varepsilon T_0) \tag{3.12}$$

where we have included the prefactor from (3.2).

The other two contributions to the 'figure of eight' can now be dealt with more quickly. Following the same strategy as before, the second term in (3.2) can be written as

$$\int \mathrm{d}\Omega\,\langle f^2(L_{ij}f)^2\rangle=-(T_0R^{2-\varepsilon})^2S_d^{-1}\left(\sum_{l\geqslant 2}\nu_l\right)\left(\sum_{k\geqslant 2}\frac{\nu_k}{k(k+\varepsilon-1)}\right)+\mathrm{O}(\varepsilon T_0^2). \tag{3.13}$$

It does not present any difficulties to show that the second sum in (3.13) is $\mathrm{O}(\varepsilon)$, such that the whole expression is $\mathrm{O}(T_0\varepsilon)$ and can be safely neglected at two-loop level. The same statement is true for the third contribution in (3.2) which, apart from prefactors, can be written as the square of the second sum in (3.13) and is therefore $\mathrm{O}(\varepsilon^2T_0)$.

Thus we have derived the total contribution from the 'figure of eight' graph:

$$\langle \mathscr{H}_{\mathrm{int}}^{(4)}\rangle=-(T_0R^{-\varepsilon}/S_d)(1/\varepsilon+\tfrac{3}{2})+\mathrm{O}(\varepsilon T_0). \tag{3.14}$$

Its significance will be exploited once the cubic interactions have been discussed.

3.3. Cubic interactions

We have to discuss two types of cubic interaction:

$$\mathscr{H}_{\text{int}}^{(3)} = -\frac{R^{\varepsilon-3}}{T_0}\int \mathrm{d}\Omega \left[\frac{\varepsilon-2}{4}(L_{ij}f)^2 f + \binom{\varepsilon}{3} f^3\right] = -\frac{R^{\varepsilon-3}}{T_0}\int \mathrm{d}\Omega \left[\frac{2-\varepsilon}{8} f^2 L^2 f + \binom{\varepsilon}{3} f^3\right]. \tag{3.15}$$

Trivially, the average over an odd number of fields with respect to a quadratic Hamiltonian is zero. However, the square of $\mathscr{H}_{\text{int}}^{(3)}$ does give a contribution. It will be represented by the London Transport or the dumbbell graph depending on how the internal legs are connected.

Let us investigate the dumbbell graph first, with an f^3 interaction on both vertices. Recalling $G(l)$ from (2.7), the corresponding sum reads, up to various prefactors,

$$\begin{aligned}
&\sum_{l\geqslant 2;\alpha}\sum_{k\geqslant 2;\beta}\sum_{p\geqslant 2;\gamma} G(l)G(k)G(p)\int \mathrm{d}\Omega \\
&\qquad \times \int \mathrm{d}\Omega'\; Y_l^{\alpha}(\boldsymbol{\eta}) Y_l^{\alpha}(\boldsymbol{\eta}) Y_k^{\beta}(\boldsymbol{\eta}) Y_k^{\beta}(\boldsymbol{\eta}') Y_p^{\gamma}(\boldsymbol{\eta}') Y_p^{\gamma}(\boldsymbol{\eta}') \\
&\qquad = S_d^{-2}\sum_{l\geqslant 2}\sum_{k\geqslant 2;\beta}\sum_{p\geqslant 2} \nu_l(d)G(l)G(k)\nu_p(d)G(p)\int \mathrm{d}\Omega\; Y_k^{\beta}(\boldsymbol{\eta})\int \mathrm{d}\Omega'\; Y_k^{\beta}(\boldsymbol{\eta}') \\
&\qquad = 0
\end{aligned} \tag{3.16}$$

because the $k=0$ mode has been excluded and

$$\int \mathrm{d}\Omega\; Y_k^{\beta}(\boldsymbol{\eta}) = S_d^{-1/2}\delta_{\beta,0}\delta_{k,0}. \tag{3.17}$$

A derivative interaction f^2L^2f on one or both of the two vertices does not change the reasoning behind (3.16): the operator L^2 acting on a spherical harmonic simply generates its eigenvalue as additional factor in the numerator. This does not influence the integration over $\mathrm{d}\Omega$ and $\mathrm{d}\Omega'$; hence these contributions vanish as well.

This result can be understood as follows: angular momentum conservation only allows for $l=0$ on the line connecting the two vertices. But the zero mode has been excluded from all sums, and thus the contribution from this graph vanishes altogether.

Finally, we consider the London Transport graph with two f^3 interactions. All three internal lines start at one vertex and end at the other; thus we get the sum

$$\begin{aligned}
\Sigma = &\sum_{l\geqslant 2;\alpha}\sum_{k\geqslant 2;\beta}\sum_{p\geqslant 2;\gamma} G(l)G(k)G(p) \\
&\times \int \mathrm{d}\Omega \int \mathrm{d}\Omega'\; Y_l^{\alpha}(\boldsymbol{\eta}) Y_l^{\alpha}(\boldsymbol{\eta}') Y_k^{\beta}(\boldsymbol{\eta}) Y_k^{\beta}(\boldsymbol{\eta}') Y_p^{\gamma}(\boldsymbol{\eta}) Y_p^{\gamma}(\boldsymbol{\eta}') \\
&= S_d^{-3}\sum_{l\geqslant 2}\sum_{k\geqslant 2}\sum_{p\geqslant 2} G(l)G(k)G(p)\nu_l(d)\nu_k(d)\nu_p(d)[C_l^{\lambda}(1)C_k^{\lambda}(1)C_p^{\lambda}(1)]^{-1} \\
&\times S_d S_{d-1}\int_{-1}^{+1} C_l^{\lambda}(t)C_k^{\lambda}(t)C_p^{\lambda}(t)(1-t^2)^{\lambda-1/2}\,\mathrm{d}t
\end{aligned} \tag{3.18}$$

where $\lambda = \frac{1}{2}(d-2)$ and we have used (3.6).

The integral occurring in (3.18) was evaluated by Hsü (1938):

$$\int_{-1}^{+1} C_l^\lambda(t) C_k^\lambda(t) C_p^\lambda(t)(1-t^2)^{\lambda-1/2}\,\mathrm{d}t$$
$$= \frac{2^{1-2\lambda}}{[\Gamma(\lambda)]^4}\pi \frac{\Gamma(s+2\lambda)}{(s+\lambda+1)} \frac{\Gamma(s-l+\lambda)}{(s-l+1)} \frac{\Gamma(s-k+\lambda)}{\Gamma(s-k+1)} \frac{\Gamma(s-p+\lambda)}{\Gamma(s-p+1)} \Delta(l,k,p) \tag{3.19}$$

where

$$2s = l+k+p \text{ and}$$

$$\Delta(l,k,p) = \begin{cases} 1 & \text{if } l+k+p \text{ is even and } l,\ k,\ p \text{ can form the sides of a triangle,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

In three dimensions, equation (3.20) is familiar from quantum mechanics: it describes how two orbital angular momenta k, l couple to give a third, p. The prefactors in (3.19) are essentially the corresponding Clebsch–Gordan coefficients.

It appears impracticable to evaluate Σ exactly from (3.18)–(3.20). However, this is not necessary. To extract the ε dependence it is sufficient to consider (3.18) in the limit of large l, k and p. For large l we have the following:

$$\begin{aligned} &G(l) = T_0 R^{2-\varepsilon} l^{-2}[1+\mathrm{O}(\varepsilon, l^{-1}, \ldots)], \\ &\nu_l(d)/S_d = S_{d-1} l^{\varepsilon-1}[1+\mathrm{O}(\varepsilon, l^{-1}, \ldots)], \\ &\Gamma(l+\alpha)/\Gamma(l+\beta) = l^{\alpha-\beta}[1+\tfrac{1}{2}(\alpha-\beta)(\alpha+\beta-1)/l+\mathrm{O}(l^{-2})]. \end{aligned} \tag{3.21}$$

Inserting (3.21) into (3.18) and (3.19) one finds after a bit of algebra:

$$\Sigma = -\frac{1}{4\pi} S_d S_{d-1} \sum_{\substack{l\geqslant 2 \\ k\geqslant 2 \\ p\geqslant 2}} l^{\varepsilon-1} G(l) k^{\varepsilon-1} G(k) p^{\varepsilon-1} G(p)$$
$$\times (lkp)^{1-2\lambda}[s(s-l)(s-k)(s-p)]^{\lambda-1}\Delta(l,k,p). \tag{3.22}$$

Here, corrections of $\mathrm{O}(\varepsilon)$ or $\mathrm{O}(l^{-1}, p^{-1}, k^{-1})$ have been neglected. We shall soon see that this is justified.

We shall now proceed as follows. Firstly, the sums in (3.22) will be replaced by integrals. This is a reasonable approximation for large values of l, k and p. Secondly, instead of working on the sphere, we investigate the London Transport graph in $(d-1)$-dimensional Euclidean space where

$$G(\boldsymbol{x}, \boldsymbol{x}') = \int_{|l|>\Lambda} \frac{\mathrm{d}^{d-1}q}{(2\pi)^{d-1}} G(q) \exp[\mathrm{i}\boldsymbol{q}(\boldsymbol{x}-\boldsymbol{x}')]$$

and

$$G(q) = T_0 q^{-2}. \tag{3.23}$$

The infrared cut-off Λ has been introduced in order to avoid infrared divergences. This corresponds to the exclusion of the $l=0$ and $l=1$ modes in the spherical theory. We shall show that the London Transport graph in Euclidean formulation can also be cast in the form (3.22), with the sums replaced by integrals. Hence we can approximate (3.18) by its counterpart in Euclidean space. Physically this means that for large angular momenta the spherical interface behaves essentially like a planar interface.

Thirdly, we find the ε dependence of (3.18) by evaluating the planar expression using standard techniques.

Replacing the sums by integrals, (3.22) becomes

$$\Sigma = -\frac{S_d}{4\pi} S_{d-1} \int_{|l|>\Lambda} \mathrm{d}l\, l^{\varepsilon-1} \int_{|k|>\Lambda} \mathrm{d}k\, k^{\varepsilon-1} \int_{|p|>\Lambda} \mathrm{d}p\, p^{\varepsilon-1} G(l)G(k)G(p)(lkp)^{1-2\lambda} \times [s(s-l)(s-k)(s-p)]^{\lambda-1}\Delta(l,k,p). \tag{3.24}$$

l, k and p can now take on non-integer values as well. In the definition (3.20) of $\Delta(l, k, p)$ we therefore keep only the triangle condition, and we drop the requirement that the sum of l, k and p has to be even.

In Euclidean space, the London Transport graph corresponds to

$$\begin{aligned} I &= \int \mathrm{d}^{d-1}x' \int \mathrm{d}^{d-1}x \int_\Lambda \frac{\mathrm{d}^{d-1}l}{(2\pi)^{d-1}} \int_\Lambda \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \int_\Lambda \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} G(l)G(k)G(p) \\ &\quad \times \exp[\mathrm{i}(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{l})(\boldsymbol{x}-\boldsymbol{x}')] \\ &= V \int \mathrm{d}^{d-1}x \int_\Lambda \frac{\mathrm{d}^{d-1}l}{(2\pi)^{d-1}} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}} \exp[\mathrm{i}(\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{l})\boldsymbol{x}] \\ &\quad \times G(l)G(k)G(p). \end{aligned} \tag{3.25}$$

Instead of integrating over $\boldsymbol{x}$ first, we begin by integrating over the angular parts of $\boldsymbol{l}$, $\boldsymbol{k}$ and $\boldsymbol{p}$. This leads to the Poisson integral (Erdélyi 1953):

$$\int \mathrm{d}\Omega_q\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\boldsymbol{x}} = S_{d-2} \int_{-1}^{+1} (1-t^2)^{\lambda-1}\, \mathrm{e}^{\mathrm{i}qxt} = S_{d-2}\Gamma(\lambda) J_{\lambda-1/2}(qx)\sqrt{\pi}(\tfrac{1}{2}qx)^{1/2-\lambda} \tag{3.26}$$

where $J_{\lambda-1/2}(qx)$ is a Bessel function of order $\lambda - \frac{1}{2}$.

Now the resulting integral over $\boldsymbol{x}$ can be performed. Its angular part is trivial, hence

$$\begin{aligned} &S_{d-1} \int \mathrm{d}x\, J_{\lambda-1/2}(lx) J_{\lambda-1/2}(kx) J_{\lambda-1/2}(px) x^{3/2-\lambda} \\ &\quad = S_{d-1} \frac{2^{\lambda-3/2}}{\sqrt{\pi}\Gamma(\lambda)} (lkp)^{1/2-\lambda} A^{2\lambda-2} \Delta(l,k,p) \end{aligned} \tag{3.27}$$

where again

$$\Delta(l,k,p) = \begin{cases} 1 & \text{if } l, k, p \text{ form the sides of a triangle,} \\ 0 & \text{otherwise,} \end{cases}$$

and A is the area of the triangle formed by l, k and p (Watson 1922).

If the perimeter of the triangle is given by $2s$, one has

$$A = [s(s-l)(s-k)(s-p)]^{1/2}$$

so finally

$$\begin{aligned} I = -VS_{d-1} \frac{1}{8\pi} &\int_\Lambda \mathrm{d}l \int_\Lambda \mathrm{d}k \int_\Lambda \mathrm{d}p\, l^{\varepsilon-1} G(l) k^{\varepsilon-1} G(k) p^{\varepsilon-1} G(p) \\ &\times (lkp)^{1-2\lambda} [s(s-l)(s-k)(s-p)]^{\lambda-1} \Delta(l,k,p). \end{aligned} \tag{3.28}$$

which agrees up to a trivial prefactor of O(1) with (3.24)

It is now obvious that (3.28) and (3.25) must have the same ε-dependence.

Also, in the case that we have an f^2L^2f interaction on one or both of the two vertices, we note that for large l

$$L^2 Y_l^\alpha(\boldsymbol{\eta}) = -2l^2 Y_l^\alpha(\boldsymbol{\eta}) \tag{3.29}$$

which has to be replaced by

$$\nabla^2 e^{i\mathbf{q}\mathbf{x}} = -q^2 e^{i\mathbf{q}\mathbf{x}} \tag{3.30}$$

i.e. an interaction $f^2\nabla^2 f$, in the planar formulation; and again we derive the required correspondence between the two versions.

It is straightforward to calculate the London Transport graph in the planar version with interactions f^3 or $f^2\nabla^2 f$. Recalling that the whole graph carries a factor T_0, we find that it is at most $O(\varepsilon T_0)$, and the corrections neglected in (3.22) were even higher order.

This concludes the discussion of two-loop graphs. We have found only one contribution to the two-loop partition function, arising from the figure of eight. This contribution exponentiates according to the well known theorem for connected vacuum graphs. Hence, with (2.19) and (3.14), the full two-loop partition function reads:

$$\begin{aligned} Z_1 = S_d^{(d+1)/2} d^{-d/2} \int \mathrm{d}R \int \mathrm{d}^d x \left(\frac{R^{\varepsilon-2}}{2\pi T_0}\right)^{(2+\varepsilon)/2} \\ \times \exp\left(-\frac{S_d R^\varepsilon}{T_0} - \frac{2}{\varepsilon\Gamma(d)} + \frac{T_0 R^{-\varepsilon}}{\varepsilon S_d} + \frac{\varepsilon}{\Gamma(d)}(C+2\gamma_1) + \frac{3}{2}\frac{T_0 R^{-\varepsilon}}{S_d}\right) \\ \times [1 + O(\varepsilon^2, \varepsilon T_0, T_0^2)]. \end{aligned} \tag{3.31}$$

This result is unsatisfactory for two reasons. Firstly, for $\varepsilon \to 0$, the effective expansion parameter $T_0 R^{-\varepsilon}$ becomes unboundedly large for $R \to 0$. This means that we cannot handle small droplets on the basis of (3.31) although we know that they are important close to the critical point. Secondly, the ε-divergences in the exponential have to be removed. In § 4 we shall see that the renormalisation of (3.31) takes care of both problems.

4. Renormalisation

We adopt a minimal subtraction scheme: the renormalised coupling T is defined as a power series in the bare dimensionless coupling $T_0 R^{-\varepsilon}$ such that the coefficients are pure poles in ε only and absorb all the divergences. Hence we write

$$[T_0 R^{-\varepsilon}]^{-1} = T^{-1} + a + bT + O(T^2) \tag{4.1}$$

and insert this into the partition function (3.31).

Comparing orders in T one finds that in order to subtract the divergences one needs

$$a = -1/\varepsilon, \qquad b = -1/4\varepsilon. \tag{4.2}$$

Note that the prefactor $(T_0 R^{-\varepsilon})$ in front of the exponential in (3.31) is now essential for the renormalisation, whereas it did not contribute at the one-loop level.

Inserting (4.2) into (4.1) and inverting the expression yields

$$T_0 R^{-\varepsilon} = T + (1/\varepsilon)T^2 + (1/4\varepsilon + 1/\varepsilon^2)T^3 + O(T^4). \tag{4.3}$$

This gives the β-function

$$\beta(T) = R\,\partial T/\partial R|_{T_0} = -\varepsilon T + T^2 + \tfrac{1}{2}T^3 + O(T^4) \tag{4.4}$$

and this result is indeed identical with the β-function of the planar interface. Hence we confirm the assumption made by BW and close the loophole in their work.

The renormalisation equation (4.4) has two fixed points: $T=0$ which is infrared stable: $T \to 0$ as $R \to \infty$; and $T_c = \varepsilon - \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$ which is ultraviolet stable: $T \to T_c$ as $R \to 0$. The existence of the second fixed point solves the first problem mentioned with respect to (3.31). If now $R \to 0$, the renormalised coupling T approaches the finite value T_c.

The second problem, concerned with the singularities in ε occurring in (3.31) is also solved by the renormalisation of $T_0 R^{-\varepsilon}$ as can be seen by inserting (4.3) into (3.31):

$$\begin{aligned} Z_1 = (S_d/2\pi)^{(d+1)/2} d^{-d/2} V \int \mathrm{d}R\, R^{-(d+1)} T^{-(\varepsilon+2)/2} \\ \times \exp\left(-\frac{S_d}{T} + \frac{S_d}{\varepsilon} - \frac{2}{\varepsilon\Gamma(d)} + \frac{T}{\varepsilon}\left(\frac{S_d}{4} + \frac{1}{S_d} - 1\right)\right. \\ \left. + \tfrac{1}{4}T + \varepsilon(C + 2\gamma_1)\right)[1 + O(\varepsilon^2, \varepsilon T, T^2)]. \end{aligned} \tag{4.5}$$

Here the integration over x has been carried out to give the total available volume V. It is obvious that the ε-divergences in the exponential cancel. Let us now introduce the correlation length ξ as the R-independent length scale of the theory:

$$\xi = R \exp\left(-\int^{T(R)} \mathrm{d}T'/\beta(T') \right) \tag{4.6}$$

hence

$$T(R)[1 - T(R)/T_c]^{-\varepsilon\nu}[1 + O(\varepsilon T)] = C_0(\xi/R)^{\varepsilon} \tag{4.7}$$

where

$$\nu^{-1} = \beta'(T_c) = \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3) \tag{4.8}$$

and C_0 is an integration constant.

Clearly, $T(R)$ only depends on the ratio (R/ξ), and therefore the partition function can be written in the form

$$Z_1 = V \int \mathrm{d}R\, (S_d R^d/d)^{-1} R^{-1} \tilde{\psi}(R/\xi) \tag{4.9}$$

where

$$\begin{aligned} \tilde{\psi}(R/\xi) = (S_d/2\pi)^{(d+1)/2} S_d d^{-d/2-1} [T(R)]^{-(\varepsilon+2)/2} \\ \times \exp\left[-\frac{S_d}{T(R)} + \frac{S_d}{\varepsilon} - \frac{2}{\varepsilon\Gamma(d)} + \frac{T(R)}{\varepsilon}\left(\frac{S_d}{4} + \frac{1}{S_d} - 1\right)\right. \\ \left. + \tfrac{1}{4}T(R) + \varepsilon(C + 2\gamma_1)\right][1 + O(\varepsilon^2, \varepsilon T(R), T^2(R))]. \end{aligned} \tag{4.10}$$

The function $\tilde{\psi}(R/\xi)$ plays an important role for the construction of the multi-droplet ensemble as was discussed in detail by BW.

In particular, its limit for small arguments

$$\psi_0 = \lim_{z \to 0} \tilde{\psi}(z) \tag{4.11}$$

determines essential physical quantities: for example, the magnetisation exponent β is given by $\beta = 2\psi_0 \nu$; or, more generally, in the droplet formulation of the q-state Potts model (Schmittmann 1982) one has $\beta = q\psi_0 \nu$.

Also, the critical droplet density ρ_c is proportional to the parameter ψ_0 whose smallness therefore controls the validity of the dilute surfaces approximation made by BW.

In order to obtain an expression for ψ_0, we recall that if $R/\xi \to 0$ the renormalised temperature $T(R)$ approaches T_c, hence ψ_0 is given by expression (4.10), but with $T(R)$ replaced by T_c.

Naively, one would assume that by inserting the *one-loop* expression for T_c one would get a correct *one-loop* expression for ψ_0. However, in order to obtain the exponential correct to O(1) one needs the $O(\varepsilon^2)$ corrections in T_c. This is a reflection of the fact that the first term in the coupling constant expansion is proportional to T^{-1}.

By giving the correct two-loop renormalisation of the single droplet partition function we have amended this shortfall of the one-loop calculation, and the parameter ψ_0, to one loop, is given by

$$\psi_0 = (2/\pi)\varepsilon^{-(\varepsilon+2)/2} \exp(-1-2\gamma-2/\varepsilon) \tag{4.12}$$

where γ is Euler's constant.

One should note, however, that in order to get the two-loop expression for ψ_0 the temperature renormalisation is required to three loops etc.

To conclude let us summarise our results.

As the source of divergences in two-loop graphs we identified contractions of different angular momentum operators as opposed to contractions of one angular momentum operator with itself. The tensorial character of these mixed contractions produces the singularities in ε which then have to be renormalised. We expect these features to be essential for higher-order graphs also.

Similar tensorial contractions can be found in the corresponding graphs for the planar interface, and there are apparent analogies between the two theories which persist even in calculational details.

As a consequence of the close relationship between the planar and the spherical interface we find the same two-loop β-function for both theories, hence also the same critical temperature T_c and the same correlation length exponent ν. Thus the assumption of BW is made redundant, and the discussion of single and multi-droplet properties is put on a more systematic footing.

Acknowledgments

I am very grateful to D J Wallace, A D Bruce and A J McKane for many helpful discussions. I also thank the University of Edinburgh and the Studienstiftung des deutschen Volkes for financial support.

References

Andreev A F 1963 *Zh. Eksp. Teor. Fiz.* **45** 2064 (transl. 1964 *Sov. Phys.-JETP* **18** 1415)
Becker R and Döring W 1935 *Ann. Phys., Lpz* **24** 719
Binder K 1976 *Ann. Phys., NY* **98** 390
Binder K and Stauffer D 1976 *Adv. Phys.* **25** 343
Bruce A D and Wallace D J 1981 *Phys. Rev. Lett.* **47** 1754
—— 1983 *J. Phys. A: Math. Gen.* **16** 1721
Cahn J W and Hilliard J E 1958 *J. Chem. Phys.* **28** 258
—— 1959 *J. Chem. Phys.* **31** 688
Domb C 1976 *J. Phys. A: Math. Gen.* **9** 283
Drummond I T 1975 *Nucl. Phys.* B **94** 115
Drummond I T and Shore G M 1979 *Ann. Phys., NY* **121** 204
Erdélyi A (ed) 1953 *Higher Transcendental Functions* vol 1 and 2 (New York: McGraw-Hill)
Fisher M E 1967 *Physics* **3** 255
Günther N J, Nicole D A and Wallace D J 1980 *J. Phys. A: Math. Gen.* **13** 1755
Hsü H Y 1938 *Duke Math. J.* **IV** 374
Kadanoff L P 1976 in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green vol Va (New York: Academic)
Langer J S 1967 *Ann. Phys., NY* **41** 108
McKane A J and Wallace D J 1978 *J. Phys. A: Math. Gen.* **11** 2285
Schmittmann B 1982 *J. Phys. A: Math. Gen.* **15** 3571
't Hooft G 1973 *Nucl. Phys.* B **61** 455
't Hooft G and Veltman M 1972 *Nucl. Phys.* B **44** 189
Wallace D J and Zia R K P 1979 *Phys. Rev. Lett.* **43** 808
Watson G N 1922 *A treatise on the theory of Bessel functions* (Cambridge: CUP) (2nd edn 1944)
Widom B 1972 in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green, vol II (New York: Academic)